Fluctuation theory and (very) early statistical energy analysis (SEA) (L)

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The equivalence between a noise source in combination with a resistance, and a thermal bath, is a part of fluctuation theory. This observation was the motivation for the author's 1960 calculation of the power flow between modes of vibration excited by random noise sources. The resulting relation between power flow and uncoupled energy of vibration is therefore more than a thermal analogy; it is an actual representation of two thermal baths in contact. In a sense, the answer was known before the calculation was performed; the power flow had to be proportional to the differences in modal energy. The calculation merely determined the thermal conductivity of the connection. The relations for groups of modes in contact and the proportionality between power flow and actual modal energy came later. © *2003 Acoustical Society of America.* [DOI: 10.1121/1.1567274]

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A particle in Brownian motion is bombarded equally by impacts from all sides by the thermal bath in which it resides. But if it has a drift velocity, then the impacts from the direction it is moving toward will be a bit stronger on average, leading to a resistance to its motion in that direction. There is therefore a relation between the random impacts (spectral density S_f) and the resistance to motion R. That relation is quantified if we require the energy of each degree of freedom of the particle to reach thermal equilibrium ($\frac{1}{2}kT$) with the bath (k is Boltzmann's constant, T is the absolute temperature). The result is that the spectral density (in cyclic frequency or Hz) be[1] $S_f = 4RkT$. For a resonant mode with two degrees of freedom (displacement and velocity, current and voltage, pressure and velocity potential, etc.), the equilibrium energy is $E_m = kT = S_f/4R$. (Note that engineers call a resonator a ''single degree of freedom'' device, but a physicist refers to its two degrees of freedom, each entering quadratically into the Hamiltonian.)

In 1959, the author went to the University of Manchester to work on turbulence theory, with an intention to use fluctuation theory to determine the equilibrium distribution of energy among the ''modes'' of turbulence. Such a mode is illustrated in Fig. 1, a single dof system with a broad range of frequency dependence. The excitation of such a mode is its highly nonlinear interaction with the other modes. There had been some success in estimating the turbulence spectrum for homogeneous isotropic turbulence and the hope was that fluctuation theory could be used to both broaden and focus the equilibrium spectral estimates.

Although there may still be value in the idea, I was unable to make headway with it. At the University of Minnesota, I had become interested in the modal density of structures, and that became the second or fallback focus of the Manchester work. Modal resonances had some obvious advantages; the response was narrow band and the dynamics were linear so that different frequency bands were isolated as can be seen in Fig. 1. Ordinary fluctuation theory says that all resonant modes in the system, up to the Planck limit, have the same energy of kT. But if the system is linear, one can have different frequency bands at ''different temperatures,'' and still use the concept of modal energy as the driving potential for energy flow.

The diagram in Fig. 2 illustrates how the idea was applied. Two thermal baths at different temperatures interact with attached resonators (the example uses an electrical resonant circuit). If the resonators are coupled (by magnetic flux or mutual inductance for example), we would be astonished if the power flow between them were *not* proportional to the difference in the temperatures of the bath or their uncoupled energies. Parenthetically, some commentators on SEA have thought this is a simplifying assumption. It actually makes the problem a bit harder because reciprocity cannot be invoked to aid the calculation when the resonators are uncoupled.

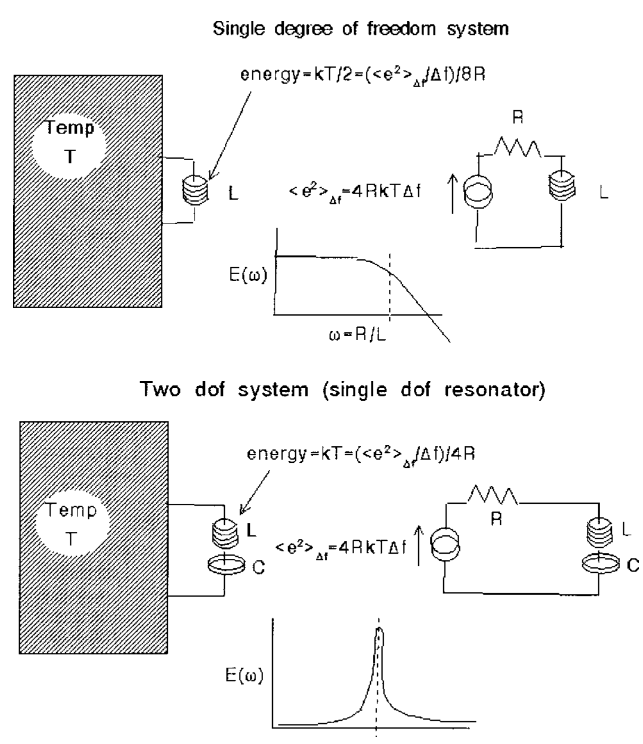


FIG. 1. One- and two-degree-of-freedom systems interacting with a thermal bath.

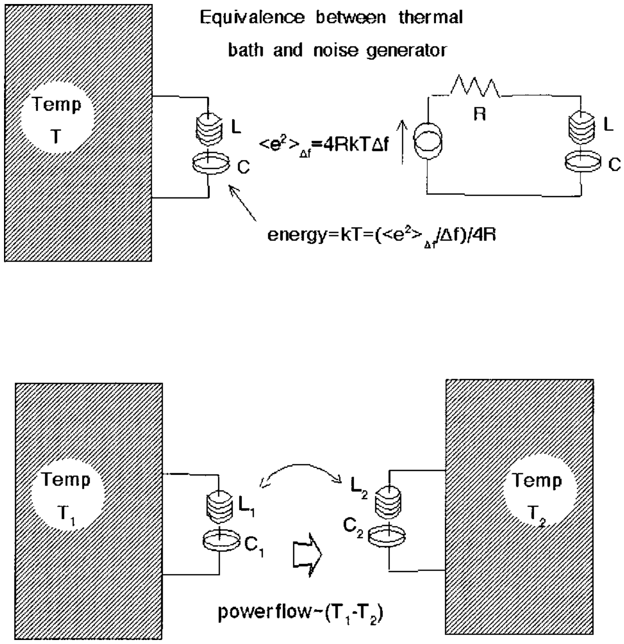


FIG. 2. Two coupled resonators at different temperatures.

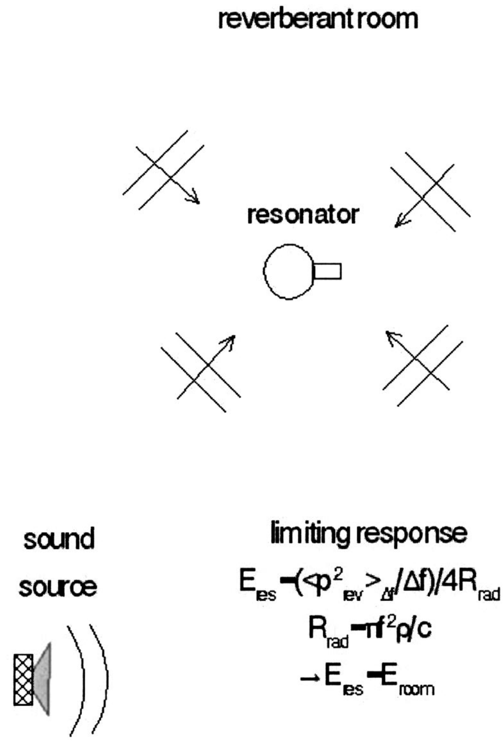
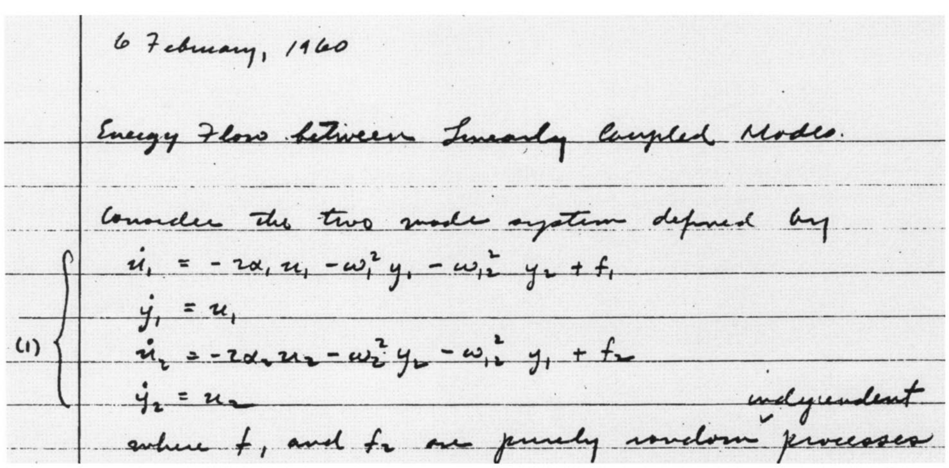


FIG. 4. Resonator in a reverberant room, with the response determined by the ‘‘temperature’’ of the room modes.

The actual calculation of the power flow between modes was recorded in my notes from Manchester in February 1960, reproduced in part in Fig. 3. It is essentially a calculation for random vibration, and it gets a result for the case where one (the ‘‘second’’) resonator has a finite resistance but no random excitation. From above, $S_f/4kT \rightarrow R$ as $S_f \rightarrow 0$ and $T \rightarrow 0$, so that the second unexcited resonator is at ‘‘absolute zero’’ when it does not have noise excitation. This is a common situation in acoustical applications. The result showed that as the resistance of the second resonator vanishes, that resonator comes to energy equilibrium with the resonator that is excited by a random force. Lyon and Maidanik[2] later published this work with additions in the *Journal of the Acoustical Society of America*.

When I joined BBN in August of 1960 I became aware that Preston Smith had carried out a calculation of the interaction of a mode with a reverberant and diffuse sound field as shown in Fig. 4. It was generally assumed at the time that a sound field of noise was equivalent to a random force excitation, but the response to such a force diverges as the damping of the resonator vanishes. Smith[3] had found that in fact that the response reached a limit when the coupling represented by the radiation damping became larger than the internal damping of the mode. It was the calculation of a moment to show that this limit was the fluctuation theory value of $E_m = S_p(f)/4R_{\mathrm{rad}}$, where S_p is the spectral density of the pressure field and $R_{\mathrm{rad}} = \pi f^2 \rho / c$ is the point source acoustic radiation resistance.

It is not at all strange that a diffuse reverberant sound field should behave like a thermal bath, since in the audible frequency range, the thermal degrees of freedom of a room are in fact its acoustical resonances. So when we excite a room with a band of noise, we are making a limited number of room modes very hot. Thankfully, the system is nearly


6 February, 1960

Energy Flow between Linearly Coupled Modes.

Consider the two mode system defined by

$$(1)\quad \begin{cases} \dot{u}_1 = -2\alpha_1 u_1 - \omega_1^2 y_1 - \omega_{12}^2 y_2 + f_1 \\ \dot{y}_1 = u_1 \\ \dot{u}_2 = -2\alpha_2 u_2 - \omega_2^2 y_2 - \omega_{12}^2 y_1 + f_2 \\ \dot{y}_2 = u_2 \end{cases}$$

where f_1 and f_2 are purely independent random processes

FIG. 3. Excerpt from R. H. Lyon’s notebook, written in Manchester in 1960.

linear, so all of the other modes are very weakly connected and they stay cool!

In the early and mid-1960s SEA acquired a name and the applications and extensions of the theory came fast and furious. Those of us lucky to have been participants in the process feel fortunate to have been a part of that group, in that place, and at that time.

[1] J. I. Lawson and G. E. Uhlenbeck, eds., *Threshold Signals (MIT Radiation Laboratory Series*, Vol. 24 (published by McGraw-Hill, New York, 1950; reprinted by Boston Technical, Boston, 1964; reprinted by Dover, New York, 1963), Secs. 4-1–4-5.

[2] R. H. Lyon and G. Maidanik, ''Power flow between linearly coupled oscillators,'' J. Acoust. Soc. Am. **34**, 632–639 (1962).

[3] P. W. Smith, Jr., ''Response and radiation of structural modes excited by sound,'' J. Acoust. Soc. Am. **34**, 640–647 (1962).